

Exact solutions of equations for the Burgers hierarchy

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Abstract

Some classes of the rational, periodic and solitary wave solutions for the Burgers hierarchy are presented. The solutions for this hierarchy are obtained by using the generalized Cole - Hopf transformation.

1 Introduction

The Burgers hierarchy is well known family of nonlinear evolution equations. This hierarchy can be written in the form

$$u_t + \alpha \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + u \right)^n u = 0, \quad n = 0, 1, 2, \dots, \quad (1)$$

At $n = 1$ Eq. (1) is the Burgers equation

$$u_t + 2\alpha u u_x + \alpha u_{xx} = 0. \quad (2)$$

Eq. (2) was firstly introduced in [1]. It's well known that the Burgers equation can be linearized by the Cole—Hopf transformation [2, 3]. Exact solutions of Eq.(2) were discussed in many papers(see for example [4–7]).

In the case $n = 2$ from Eq. (1) we have the Sharma - Tasso - Olver (STO) equation

$$u_t + \alpha u_{xxx} + 3\alpha u_x^2 + 3\alpha u u_{xx} + 3\alpha u^2 u_x = 0. \quad (3)$$

The STO equation was derived in [8,9]. Some exact solutions of this equation was obtained in [10–16].

At $n = 3$ and $n = 4$ we have the following fourth and fifth order partial differential equations

$$u_t + \alpha u_{xxxx} + 10\alpha u_x u_{xx} + 4\alpha u u_{xxx} + 12\alpha u u_x^2 + 6\alpha u^2 u_{xx} + 4\alpha u^3 u_x = 0, \quad (4)$$

$$u_t + \alpha u_{xxxxx} + 10\alpha u_{xx}^2 + 15\alpha u_x u_{xxx} + 5\alpha u u_{xxxx} + 15\alpha u_x^3 + 50\alpha u u_x u_{xx} + 10\alpha u^2 u_{xxx} + 30\alpha u^2 u_x^2 + 10\alpha u^3 u_{xx} + 5\alpha u^4 u_x = 0. \quad (5)$$

In this paper we present the generalized Cole—Hopf transformation which we use for finding different types of exact solutions: the solitary wave solutions, the periodic solutions and the rational solutions. The advantage of our approach is that we can find the exact solutions for whole Burgers hierarchy. We can construct them without using the traveling wave. This fact allows us to obtain solutions of different types.

2 Generalized Cole — Hopf transformation for solutions of the Burgers hierarchy

Eq. (1) can be linearized by the Cole—Hopf transformation [9, 17, 18]

$$u = \frac{\Psi_x}{\Psi}, \quad \Psi = \Psi(x, t) \quad (6)$$

Taking this transformation into account, we have [18]

$$u_t + \alpha \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + u \right)^n u = \frac{\partial}{\partial x} \left(\frac{\Psi_t + \alpha \Psi_{n+1,x}}{\Psi} \right), \quad (7)$$

where $\Psi_{n,x}$ - n-th derivative of Ψ with respect to x .

Exact solutions of the Burgers equation can be obtained by using a generalization of the Cole—Hopf transformation [17–23]. This transformation can be written as

$$u = \frac{F_x}{F} + F, \quad F = F(x, t) \quad (8)$$

where $F(x, t)$ satisfies the Burgers equation. Let us show that transformation (8) is valid for all hierarchy (1). First of all, we prove the following lemma.

Lemma 1 The following identity takes place

$$\left(\frac{\partial}{\partial x} + \frac{\Psi_{xx}}{\Psi_x} \right)^n \frac{\Psi_{xx}}{\Psi_x} = \frac{\Psi_{n+2,x}}{\Psi_x}, \quad (9)$$

where $\Psi_{n,x}$ is n-th derivative of Ψ with respect to x .

Proof. Let us apply the method of mathematical induction. When $n = 1$ we get

$$\left(\frac{\partial}{\partial x} + \frac{\Psi_{xx}}{\Psi_x} \right) \frac{\Psi_{xx}}{\Psi_x} = \frac{\Psi_{xxx}}{\Psi_x} \quad (10)$$

At $n = 2$ we have

$$\left(\frac{\partial}{\partial x} + \frac{\Psi_{xx}}{\Psi_x} \right)^2 \frac{\Psi_{xx}}{\Psi_x} = \left(\frac{\partial}{\partial x} + \frac{\Psi_{xx}}{\Psi_x} \right) \frac{\Psi_{xxx}}{\Psi_x} = \frac{\Psi_{xxxx}}{\Psi_x} \quad (11)$$

By the induction, assuming $n = k - 1$, we obtain

$$\left(\frac{\partial}{\partial x} + \frac{\Psi_{xx}}{\Psi_x} \right)^{k-1} \frac{\Psi_{xx}}{\Psi_x} = \frac{\Psi_{k+1,x}}{\Psi_x} \quad (12)$$

Finally, when $n = k$ we have

$$\begin{aligned} \left(\frac{\partial}{\partial x} + \frac{\Psi_{xx}}{\Psi_x} \right)^k \frac{\Psi_{xx}}{\Psi_x} &= \left(\frac{\partial}{\partial x} + \frac{\Psi_{xx}}{\Psi_x} \right) \left(\frac{\partial}{\partial x} + \frac{\Psi_{xx}}{\Psi_x} \right)^{k-1} \frac{\Psi_{xx}}{\Psi_x} = \\ &= \left(\frac{\partial}{\partial x} + \frac{\Psi_{xx}}{\Psi_x} \right) \frac{\Psi_{k+1,x}}{\Psi_x} = \frac{\Psi_{k+2,x}}{\Psi_x} \end{aligned} \quad (13)$$

This equality completes the proof. \square

Theorem 1 Let $F(x, t)$ be a solution of Eq. (1). Then

$$u = \frac{F_x}{F} + F \quad (14)$$

is the solution of the Burgers hierarchy (1).

Proof. Using the Cole-Hopf transformation (6), we obtain

$$\begin{aligned} F_t &= \frac{\partial}{\partial x} \left(\frac{\Psi_t}{\Psi} \right) \\ \frac{F_t}{F} &= \frac{\Psi_{x,t}}{\Psi_x} - \frac{\Psi_t}{\Psi} \\ \frac{F_x + F^2}{F} &= \frac{\Psi_{xx}}{\Psi_x} \end{aligned} \quad (15)$$

Substituting transformation (14) into hierarchy (1) and taking the **Lemma 1** and Eq. (15) into account we have following set of equalities

$$\frac{F_{x,t}}{F} - \frac{F_x F_t}{F^2} + F_t + \alpha \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + \frac{F_x}{F} + F \right)^n \left(\frac{F_x}{F} + F \right) = \quad (16)$$

$$\begin{aligned} &= \frac{\partial}{\partial x} \left(\frac{F_t}{F} \right) + F_t + \alpha \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + \frac{F_x + F^2}{F} \right)^n \left(\frac{F_x + F^2}{F} \right) = \\ &= \frac{\partial}{\partial x} \left(\frac{\Psi_{x,t}}{\Psi_x} - \frac{\Psi_t}{\Psi} \right) + \frac{\partial}{\partial x} \left(\frac{\Psi_t}{\Psi} \right) + \alpha \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + \frac{\Psi_{xx}}{\Psi_x} \right)^n \left(\frac{\Psi_{xx}}{\Psi_x} \right) = \quad (17) \end{aligned}$$

$$= \frac{\partial}{\partial x} \left(\frac{\Psi_{x,t}}{\Psi_x} + \alpha \left(\frac{\partial}{\partial x} + \frac{\Psi_{xx}}{\Psi_x} \right)^n \left(\frac{\Psi_{xx}}{\Psi_x} \right) \right) = \quad (18)$$

Thus, we have that if $F(x, t)$ satisfies equation (1) then $u(x, t)$ by formula (14), is solution of (1) as well. \square

3 Solitary wave solutions of the Burgers hierarchy

Let us show that the Burgers hierarchy has the solution in the form

$$U_l^{(n+1, N)}(x, t) = \frac{\sum_{j=1}^N k_j^l \exp \left(k_j x - \alpha k_j^{n+1} t - x_0^{(j)} \right)}{\sum_{j=1}^N k_j^{l-1} \exp \left(k_j x - \alpha k_j^{n+1} t - x_0^{(j)} \right)}, \quad (19)$$

$$(j = 1, 2, \dots, N), \quad (n, l = 1, 2, \dots)$$

where k_j and $x_0^{(j)}$ are arbitrary constants.

This result follows from the theorem.

Theorem. Let

$$U_0 = \frac{\psi_x}{\psi}, \quad (20)$$

be a solution of the Burgers hierarchy. Then

$$U_{k+1} = \frac{\psi_{k+1,x}}{\psi_{k,x}}, \quad \psi_{k,x} = \frac{\partial^k \psi}{\partial x^k}, \quad (21)$$

is a solution of the Burgers hierarchy as well.

Proof. This theorem follows from the generalized transformation (14) for the solution of the hierarchy (1). Let

$$U_0 = F(x, t) \quad (22)$$

be the solution of the hierarchy (1) equation, then

$$U_1 = \frac{F_x + F^2}{F} \quad (23)$$

is also the solution of the Burgers hierarchy by the generalized transformation for the solution of the hierarchy (1).

Formulae (22) and (23) can be written in the form

$$U_0 = \frac{\psi_x}{\psi}, \quad U_1 = \frac{\psi_{xx}}{\psi_x}, \quad (24)$$

Assume that

$$U_m = \frac{\psi_{m,x}}{\psi_{m-1,x}}, \quad \psi_{m,x} = \frac{\partial^m \psi}{\partial x^m}, \quad (25)$$

is the solution of the hierarchy (1) and substituting U_m into the generalized transformation (14) we obtain that

$$U_{m+1} = \frac{U_{m,x} + U_m^2}{U_m} = \frac{\psi_{m+1,x}}{\psi_{m,x}} \quad (26)$$

is a solution of hierarchy (1).

This equality completes the proof. \square

This theorem allows us to have the solutions of the Burgers hierarchy in the form (19).

It is obvious, that function

$$\psi^{n+1}(x, t) = \sum_{j=0}^N \exp(z_j), \quad z_j = k_j x - \alpha k_j^{n+1} t - x_0^{(j)} \quad (27)$$

is the solution of the

$$\psi_t + \alpha \psi_{n+1,x} = 0 \quad (28)$$

By the Cole — Hopf transformation (6) we have the solution of the Burgers hierarchy in the form

$$U^{(n+1, N)} = \frac{\sum_{j=0}^N k_j \exp(z_j)}{\sum_{j=0}^N \exp(z_j)}, \quad z_j = k_j x - \alpha k_j^{n+1} t - x_0^{(j)} \quad (29)$$

Taking the theorem into account we have the solution of the hierarchy (1) in the form (19).

Let us present some examples. When $N = 2$, $n = l = 1$ we have the solution of the Burgers equation

$$U_1^{(2,2)} = \frac{k_1 \exp(z_1) + k_2 \exp(z_2)}{\exp(z_1) + \exp(z_2)}, \quad (30)$$

$$z_j = k_j x - \alpha k_j^2 t - x_0^{(j)}, \quad (j = 1, 2)$$

In the case $N = 2$, $n = l = 2$ we have the solution of the Sharma—Tasso—Olver equation in the form

$$U_2^{(3,2)} = \frac{k_1^2 \exp(z_1) + k_2^2 \exp(z_2)}{k_1 \exp(z_1) + k_2 \exp(z_2)}, \quad (31)$$

$$z_j = k_j x - \alpha k_j^3 t - x_0^{(j)}, \quad (j = 1, 2)$$

When $N = 3$, $n = 2$ and $l = 5$ we obtain the following solution of the Sharma—Tasso—Olver equation

$$U_5^{(3,3)} = \frac{k_1^5 \exp(z_1) + k_2^5 \exp(z_2) + k_3^5 \exp(z_3)}{k_1^4 \exp(z_1) + k_2^4 \exp(z_2) + k_3^4 \exp(z_3)}, \quad (32)$$

$$z_j = k_j x - \alpha k_j^3 t - x_0^{(j)}, \quad (j = 1, 2, 3)$$

In the case $N = 2$, $n = 3$ and $l = 3$ we have solitary wave solution for the Eq. (4)

$$U_2^{(4,2)} = \frac{k_1^3 \exp(z_1) + k_2^3 \exp(z_2)}{k_1^2 \exp(z_1) + k_2^2 \exp(z_2)}, \quad (33)$$

$$z_j = k_j x - \alpha k_j^4 t - x_0^{(j)}, \quad (j = 1, 2)$$

We can see that Eq. (28) is linear and has polynomial solutions. Thus, we can present its solution in the form

$$\begin{aligned} \Psi^{n+1}(x, t) &= \sum_{i=0}^I C_i x^i + \sum_{j=0}^N e^{z_j}, \\ z_j &= k_j x - \alpha k_j^{n+1} t - x_0^{(j)}, \\ (j &= 0, 1, 2, \dots, N), \quad (n = 1, 2, \dots), \quad N \in \mathbb{N}, \\ (I &= 0, 1, 2, \dots, n), \quad (i = 0, 1, 2, \dots, I). \end{aligned} \tag{34}$$

From the transformation (6) and formula (34) we have the exact solution of the hierarchy (1) in the form

$$\begin{aligned} U^{(n+1, N)}(x, t) &= \frac{\sum_{i=0}^I i C_i x^{i-1} + \sum_{j=0}^N k_j e^{z_j}}{\sum_{i=0}^I C_i x^i + \sum_{j=0}^N e^{z_j}}, \\ (j &= 0, 1, 2, \dots, N), \quad (n = 1, 2, \dots), \quad N \in \mathbb{N}, \\ (I &= 0, 1, 2, \dots, n), \quad (i = 0, 1, 2, \dots, I). \end{aligned} \tag{35}$$

For the Burgers equation ($n=1$) from (38) we obtain the solution in the form

$$U^{(2,2)}(x, t) = \frac{C_1 + k_1 e^{k_1 x - \alpha k_1^2 t - x_1} + k_2 e^{k_2 x - \alpha k_2^2 t - x_2}}{C_0 + C_1 x + e^{k_1 x - \alpha k_1^2 t - x_1} + e^{k_2 x - \alpha k_2^2 t - x_2}} \tag{36}$$

We demonstrate solution (37) when $C_0 = C_1 = k_1 = 1$, $k_2 = 2$ on Fig. 1. For the Eq. (4) from (38) we have the following solution

$$U^{(4,1)}(x, t) = \frac{C_1 + 2C_2 x + 3C_3 x^2 + k_1 e^{k_1 x - \alpha k_1^4 t - x_1}}{C_0 + C_1 x + C_2 x^2 + C_3 x^3 + e^{k_1 x - \alpha k_1^4 t - x_1}} \tag{37}$$

By analogy with solution (19) we can look for the periodic solutions of the equation for the Burgers hierarchy taking the trigonometric functions into consideration. Equation (28) has trigonometric solutions at $n + 1 = 2l + 1$

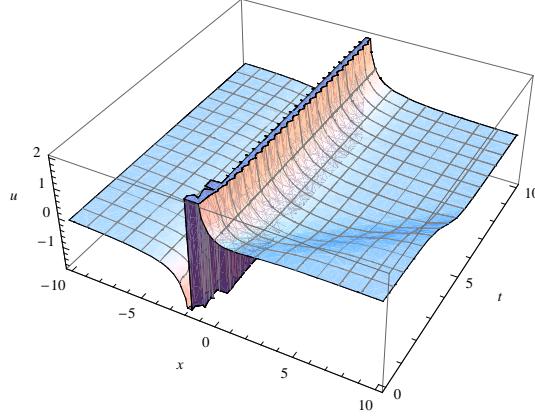


Figure 1: The solution (37) of the Burgers equation

in the form

$$\Psi^{2l+1}(x, t) = \sum_{i=0}^I C_i x^i + \sum_{m=0}^M \sin z'_m + \sum_{p=0}^P \cos z'_p + \sum_{j=0}^N e^{z_j},$$

$$z_j = k_j x - \alpha k_j^{2l+1} t - x_0^j,$$

$$z'_{m,p} = k_{m,p}^{(1,2)} x + (-1)^{l+1} \alpha (k_{m,p}^{(1,2)})^{2l+1} t - x_{(1,2)}^{(m,p)}, \quad (38)$$

$$(j = 0, 1, 2, \dots, N), \quad (l = 1, 2, \dots), \quad N \in \mathbb{N},$$

$$(I = 0, 1, 2, \dots, 2l), \quad (i = 0, 1, 2, \dots, I),$$

$$(m, p = 0, 1, 2, \dots, M), \quad M, P \in \mathbb{N}.$$

From the transformation (6) we have the exact solution of the hierarchy in the form (1)

$$U^{(2l+1)}(x, t) =$$

$$= \frac{\sum_{i=0}^I i C_i x^{i-1} + \sum_{m=0}^M k_m^{(1)} \cos z'_m - \sum_{p=0}^P k_p^{(2)} \sin z'_p + \sum_{j=0}^N k_j e^{z_j}}{\sum_{i=0}^I C_i x^i + \sum_{m=0}^M \sin z'_m + \sum_{p=0}^P \cos z'_p + \sum_{j=0}^N e^{z_j}} \quad (39)$$

For example, we can write following solution for the Sharma -Tasso -

Olver equation (l=1)

$$U^{(3)}(x, t) = \frac{k_1 e^{k_1 x - \alpha k_1^3 t - x_1} + \cos(k_2 x + \alpha k_2^3 t - x_2) k_2 - \sin(k_3 x + \alpha k_3^3 t - x_3) k_3}{C_0 + e^{k_1 x - \alpha k_1^3 t - x_1} + \sin(k_2 x + \alpha k_2^3 t - x_2) + \cos(k_3 x + \alpha k_3^3 t - x_3)} \quad (40)$$

Assuming $C_0 = 8$, $k_1 = 2.2$, $k_2 = 3$, $k_3 = 6$, $x_1 = 0.3$, $x_2 = 6$ and $x_3 = 2$ we demonstrate solution (41) on Fig. 2.

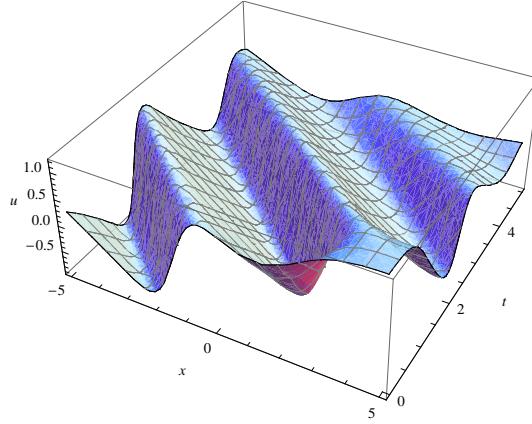


Figure 2: The solution (41) of STO equation

From formula (39) we obtain following solution for the Eq. (5) (l=2)

$$U^{(5)}(x, t) = \frac{C_0 + k_1 \cos(k_1 x + \alpha k_1^5 t - x_1) - k_2 \sin(k_2 x + \alpha k_2^5 t - x_2)}{C_0 + C_1 x + \sin(k_1 x + \alpha k_1^5 t - x_1) + \cos(k_2 x + \alpha k_2^5 t - x_2)} \quad (41)$$

Other solutions can be written using the formula (21).

4 Rational solutions of the Burgers hierarchy

Using Eq. (28) and transformations (21) and (22) we can find the rational solutions of the hierarchy (1). To obtain these solutions we use the solutions of Eq. (28) in the form

$$\psi_0(x, t) = 1, \quad \psi_1(x, t) = x, \dots, \psi_n(x, t) = x^n \quad (42)$$

Integrating $\psi_n(x, t) = x$ with respect to x we obtain $\psi_{n+1}(x, t) = x^2 + \varphi_{n+1}(t)$. Substituting $\psi_{n+1}(x, t)$ into Eq. (28) we get $\varphi_{n+1} = -(n+1)! \alpha t$. Substituting $\psi_{n+1}(x, t)$ into Eq. (28) we obtain

$$\psi_{n+1}(x, t) = x^{n+1} - (n+1)! \alpha t. \quad (43)$$

Continuing in the same way, we can obtain the solutions $\psi_q(x, t)$, $q = n + 2, n + 3, \dots$ as a result of integration of solution with respect to x . Taking these polynomial solutions of (28) into account we obtain the rational solutions of the Burgers hierarchy (1).

The polynomial solutions of (28) for $n = 2$ are the following

$$\psi_0(x, t) = 1, \quad \psi_1(x, t) = x, \quad \psi_2(x, t) = x^2, \quad (44)$$

$$\psi_3(x, t) = x^3 - 6\alpha t, \quad (45)$$

$$\psi_4(x, t) = x^4 - 24\alpha x t, \quad (46)$$

$$\psi_5(x, t) = x^5 - 60\alpha x^2 t, \quad (47)$$

$$\psi_6(x, t) = x^6 - 120\alpha x^3 t + 360\alpha^2 t^2, \quad (48)$$

$$\psi_7(x, t) = x^7 - 210\alpha x^4 t + 2520\alpha^2 x t^2, \quad (49)$$

$$\psi_8(x, t) = x^8 - 336\alpha x^5 t + 10080\alpha^2 x^2 t^2, \quad (50)$$

$$\psi_9(x, t) = x^9 - 504\alpha x^6 t + 30240\alpha^2 x^3 t^2 - 60480\alpha^3 t^3, \quad (51)$$

$$\psi_{10}(x, t) = x^{10} - 720\alpha x^7 t + 75600 x^4 \alpha^2 t^2 - 604800 \alpha^3 x t^3, \quad (52)$$

$$\psi_{11}(x, t) = x^{11} - 990\alpha x^8 t + 166320 x^5 \alpha^2 t^2 - 3326400 \alpha^3 t^3 x^2, \quad (53)$$

$$\psi_{12}(x, t) = x^{12} - 1320 \alpha x^9 t + 332640 x^6 \alpha^2 t^2 - 13305600 \alpha^3 t^3 x^3 + 19958400 \alpha^4 t^4, \quad (54)$$

Taking into account these solutions we have the rational solutions of the Sharmo—Tasso—Olver equation in the form

$$U_1(x, t) = \frac{1}{x}, \quad U_2(x, t) = \frac{2}{x} \quad (55)$$

$$U_3(x, t) = 3 \frac{x^2}{x^3 - 6 \alpha t}, \quad (56)$$

$$U_4(x, t) = 4 \frac{x^3 - 6 \alpha t}{x(x^3 - 24 \alpha t)}, \quad (57)$$

$$U_5(x, t) = 5 \frac{x^3 - 24 \alpha t}{x(x^3 - 60 \alpha t)}, \quad (58)$$

$$U_6(x, t) = 6 \frac{x^2 (x^3 - 60 \alpha t)}{x^6 - 120 \alpha x^3 t + 360 \alpha^2 t^2}, \quad (59)$$

$$U_7(x, t) = 7 \frac{x^6 - 120 \alpha x^3 t + 360 \alpha^2 t^2}{x(x^6 - 210 \alpha x^3 t + 2520 \alpha^2 t^2)}, \quad (60)$$

$$U_8(x, t) = 8 \frac{x^6 - 210 \alpha x^3 t + 2520 \alpha^2 t^2}{x(x^6 - 336 \alpha x^3 t + 10080 \alpha^2 t^2)}, \quad (61)$$

$$U_9(x, t) = 9 \frac{x^2 (x^6 - 336 \alpha x^3 t + 10080 \alpha^2 t^2)}{x^9 - 504 \alpha x^6 t + 30240 \alpha^2 x^3 t^2 - 60480 \alpha^3 t^3}, \quad (62)$$

$$U_{10}(x, t) = 10 \frac{x^9 - 504 \alpha x^6 t + 30240 \alpha^2 x^3 t^2 - 60480 \alpha^3 t^3}{x(x^9 - 720 \alpha x^6 t + 75600 \alpha^2 x^3 t^2 - 604800 \alpha^3 t^3)}, \quad (63)$$

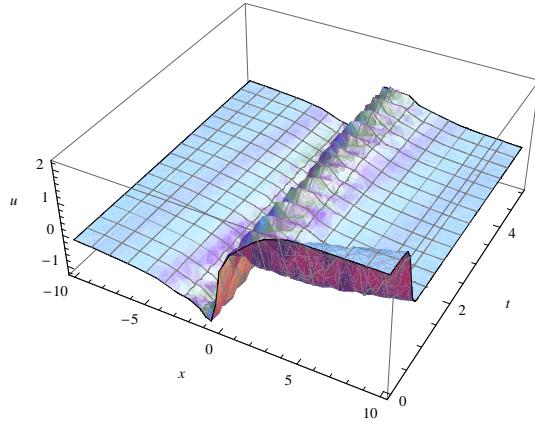


Figure 3: The solution (66) of the STO equation

$$U_{11}(x, t) = 11 \frac{x^9 - 720 \alpha x^6 t + 75600 \alpha^2 x^3 t^2 - 604800 \alpha^3 t^3}{x (x^9 - 990 \alpha x^6 t + 166320 \alpha^2 x^3 t^2 - 3326400 \alpha^3 t^3)}. \quad (64)$$

Using the solution of Eq.(28) as the sum of rational, exponential functions and, at $n = 2l$, trigonometric functions we can obtain many solutions of the hierarchy (1). In particular, at $n = 2$, taking into account solution in the form

$$\Psi(x, t) = C_2 (1 + x^2) + e^{k_1 x - \alpha k_1^3 t - x_1} + \cos(k_2 x + \alpha k_2^3 t - x_2) \quad (65)$$

we have solution of the Sharma—Tasso—Olver equation in the form

$$U(x, t) = \frac{2 C_2 x + k_1 e^{k_1 x - \alpha k_1^3 t - x_1} - \sin(k_2 x + \alpha k_2^3 t - x_2) k_3}{C_2 (1 + x^2) + e^{k_1 x - \alpha k_1^3 t - x_1} + \cos(k_2 x + \alpha k_2^3 t - x_2)} \quad (66)$$

Assuming $C_2 = 8$, $k_1 = 2.2$, $k_2 = 6$, $x_1 = 0.3$, and $x_2 = 2$ we obtain solution (66) on Fig. 3.

5 Conclusion

In this paper the generalized Cole-Hopf transformation was found for the Burgers hierarchy. We have presented some classes of the exact solutions for the Burgers hierarchy. These classes are expressed via the rational, exponential and triangular functions and as the sum of these functions.

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